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Angular Momentum Conserving Integration Scheme for Multibody System Dynamics in Lie-Group Setting

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In many engineering applications, such as satellite dynamics or various case-studies of the specific locomotion patterns in mechatronics and biomechanics, motion integrals of the system need to be conserved during numerical integration in order to reflect global physical properties of the analysed motion. Derivation of such integration schemes in Lie-group settings should be especially numerically efficient since Lie-group dynamical models operate directly on $SO(3)$ rotational group, avoiding local rotation parameters, and allowing for design of structure-preserving algorithms that respect underlying manifolds of the system dynamics. In this paper, angular momentum conserving integration scheme for rigid body rotational dynamics, based on the rotational group coadjoint action, is presented and tested through the case-study of freely spinning rigid body.

Key words:

Lie-groups, Special Orthogonal Group $SO(3)$, Structure Preserving Numerical Integration Methods, Angular Momentum Conservation, Coadjoint-preserving integration scheme

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1. Configuration space

The configuration space of an unconstrained multibody system (MBS) comprising k rigid bodies is modeled as a Lie-group $\mathcal{G} = \mathcal{R}^3 \times SO(3) \times ... \times \mathcal{R}^3 \times SO(3)$ (k copies of $\mathcal{R} \times SO(3)$) with the elements of the form $p = (\mathbf{x}_1, \mathbf{R}_1, ..., \mathbf{x}_k, \mathbf{R}_k)$. $\mathcal{G}$ is a Lie-group of the dimension $n = 6k$, where k is the number of the rigid bodies. The left multiplication in the group is given as $L_p : \mathcal{G} \to \mathcal{G}$, $\bar{p} \mapsto p \cdot \bar{p}$, where the product operation on $\mathcal{G}$ is defined by $p \cdot \bar{p} = (\mathbf{x}_1 + \bar{\mathbf{x}}_1,\ \mathbf{R}_1\bar{\mathbf{R}}_1, ..., \mathbf{x}_k + \bar{\mathbf{x}}_k, \mathbf{R}_k\bar{\mathbf{R}}_k)$ and the group identity element is $e = (\mathbf{0}_1, \mathbf{I}_1, ..., \mathbf{0}_k, \mathbf{I}_k)$.

With $\mathcal{G}$ so defined, its Lie-algebra is given as $\mathfrak{g} = \mathcal{R}^3 \times so(3) \times ... \times \mathcal{R}^3 \times so(3)$ with the elements of the form $v = (\mathbf{v}_1, \acute{\mathbf{E}}_1, ..\mathbf{v}_k, \acute{\mathbf{E}}_k)$. Furthermore, the differential ('tangent map') of the L_p is defined as $L'_p : T_e\mathcal{G} \to T_p\mathcal{G}$, $(\mathbf{v}_1, \acute{\mathbf{E}}_1, ..., \mathbf{v}_k, \acute{\mathbf{E}}_k) \mapsto (\dot{\mathbf{x}}_1, \mathbf{R}_1\acute{\mathbf{E}}_1, ..., \dot{\mathbf{x}}_k, \mathbf{R}_k\acute{\mathbf{E}}_k)$ (Terze, Müller, Zlatar, 2015a).

Alternatively, configuration space of rigid MBS can be modelled as $\mathcal{G}^x = SE(3)^k$. Although $SE(3)$ represents Lie-group of proper rigid body motion, computationally both alternatives have their advantages (Mueller, Terze, 2013).

2. Dynamical model

To formulate dynamical model of the system with the direct kinematic reconstruction on $\mathcal{G}$, we introduce constrained Boltzmann-Hamel equations introduced in the form

$$\mathbf{M}(p)\dot{\mathbf{v}} + \mathbf{C}^{\mathrm{T}}(p)\boldsymbol{\lambda} = \mathbf{Q}(p,\ \mathbf{v},\ t)\,, \tag{1a}$$

$$\dot{p} = L'_p(v)\,, \tag{1b}$$

$$\Phi(p) = 0\,, \tag{1c}$$

where $\mathbf{M}$ is $n \times n$ dimensional generalized inertia matrix, $\mathbf{v} \in \mathcal{R}^n$, $\mathbf{v} = \left[\mathbf{v}_1, \acute{\mathbf{E}}_1, ...\mathbf{v}_k,\ \acute{\mathbf{E}}_k\right]^T$ are the system velocities (k bodies are assumed), while p (system 'positions') and v (system velocities with the angular velocities expressed as $\acute{\mathbf{E}}_i \in so(3)$) are defined above. $\mathbf{Q}$ represents the external and all other forces, $\boldsymbol{\lambda} \in \mathcal{R}^m$ is the vector of Lagrange multipliers and $\mathbf{C}$ is $m \times n$ dimensional constraint Jacobian, such that $\boldsymbol{\Phi}'(v) = \mathbf{C}(p)\mathbf{v}$ where $\boldsymbol{\Phi}'$ is the differential of the constraint mapping $\Phi(p)$: $\mathcal{G} \to \mathcal{R}^m$ (Holm Darryl, 2008). With (1a) dynamic equations (Lagrangian equations of the first kind) are given and (1c) represents system kinematic constraints. The equation (1b) represents kinematic reconstruction equation that allows for determination of the system configuration p from the velocity field ν.

3. Coadjoint-preserving integration scheme

In some of the geometric schemes for MBS, integration algorithm that preserve Lie-group structure of $\mathcal{G}$ is constructed for solution of (1b), while the system dynamical equation (1a) is often discretised via 'classical' vector-space-based numerical methods. Although computationally correct, this practice does not utilize the geometrical properties of dynamics on the rotation space that can allow for the additional useful properties of the integration algorithm, such as preservation of the integrals of motion (Terze, Müller, Zlatar, 2015b). Therefore, in this paper we propose MBS geometric scheme that extends coadjoint-preserving integration method for $SO(3)$.

Indeed, rotational rigid body equation, written as Lie-Poisson system, can be given in the form of the coadjoint operator on the dual space of a Lie-algebra (Holm Darryl, 2008)

$$\dot{\mathbf{y}} = \mathrm{ad}^*_{\omega}\mathbf{y} = \hat{\mathbf{y}}\,\omega(\mathbf{y})\,, \tag{2}$$

where $\mathbf{y} \in so^*(3)$ is the body angular momentum and 'ad*' is the dual of 'ad' operator $\mathrm{ad}_{\mathbf{a}}\mathbf{b} = \hat{\mathbf{a}}\,\mathbf{b}$ (commutator in the Lie-algebra, here identified with $\mathcal{R}^3$). By knowing that the group coadjoint orbits are conserved quantities of the Lie-Poisson system, we write solution of (2) within the each integration step as (3)

$$\mathbf{y}_{n+1} = \mathrm{Ad}^*_{\exp(\tilde{\psi}(t))}\mathbf{y}_n\,, \tag{3}$$

$$\dot{\tilde{\psi}}(t) = \mathrm{dexp}^{-1}_{-\tilde{\psi}(t)}(\tilde{\omega}(\mathbf{y}))\,,\ \tilde{\psi}(0) = 0\,, \tag{4}$$

with $\mathrm{Ad}^*_{\mathbf{R}} = \mathrm{Ad}^{\mathrm{T}}_{\mathbf{R}}$, where $\tilde{\psi}(t)$ has to satisfy equation (4) (Iserles, Munthe–Kaas, Norsett, Zanna, 2000). Thus, we use vector space Runge-Kutta method of 4th order (RK-4) to integrate substitution equation (4) (Celledoni, Owren, 2003), before the angular momentum update step is performed according to (3). Furthermore, 'simultaneously' with the integration of dynamics, system kinematic reconstruction according to the equation $\dot{\mathbf{R}}(t) = \mathbf{R}(t)\tilde{\omega}$ is performed based on its assumed step-solution in the form $\mathbf{R}_{n+1} = \mathbf{R}_n \exp(\tilde{\psi}(t))$, where $\tilde{\psi}(t)$ has to satisfy substitution equation on Lie algebra (4). Actually, both integrations, i.e. rotational dynamics and kinematic reconstruction, can be done by the single RK-4 step followed by the respective updates $\mathbf{y}_{n+1}$ and $\mathbf{R}_{n+1}$, which made the proposed scheme particularly efficient. Since the angular momentum upgrade is based on the group coadjoint action (3), the integration algorithm fully preserves the body angular momentum.

4. Results and Discussion

As a numerical illustration of the algorithm, we consider the motion of freely spinning body (Krysl, Endres, 2005). The initial condition is body angular velocity $\omega_0 = \begin{bmatrix} 0.45549 & 0.82623 & 0.03476 \end{bmatrix}^T$ and inertia tensor with the diagonal elements $\mathbf{I} = \mathrm{diag}(0.9144, 1.098, 1.66)$.

A body angular velocity (expressed in the body coordinate system) and the rotation matrix $\mathbf{R} \in SO(3)$ entries along the main diagonal as well as the matrix determinant $(\mathbf{R}\mathbf{R}^T = \boldsymbol{I}, \det\mathbf{R} = +1)$ (showing that Lie-group upgrade $\mathbf{R}_{n+1} = \mathbf{R}_n \exp(\tilde{\psi}(t))$ reconstructs the body spatial motion that respect underlying rotation manifold) are shown in the Figures 1 and 2.

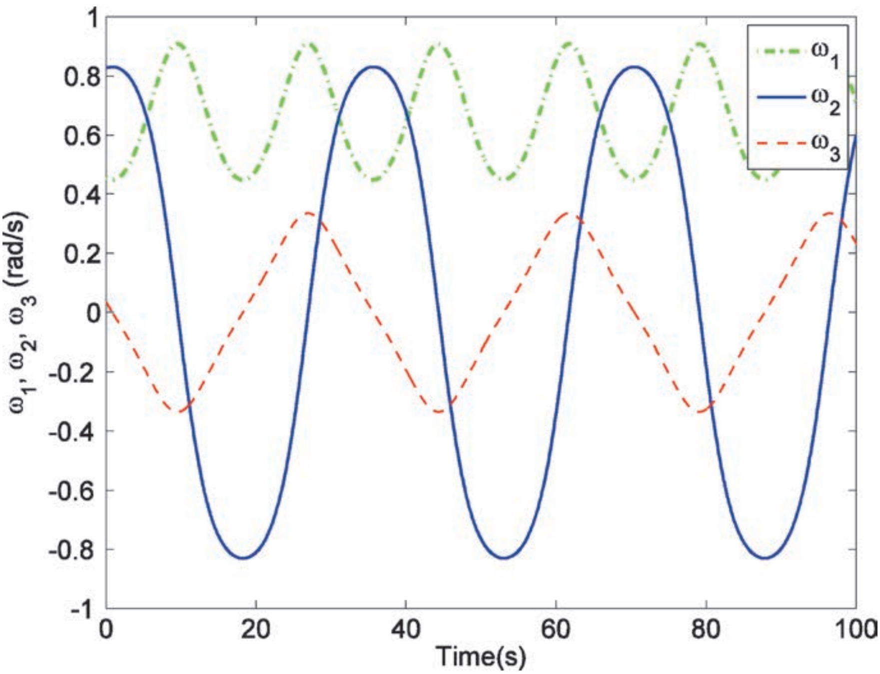

Fig. 1 – Body angular velocity.

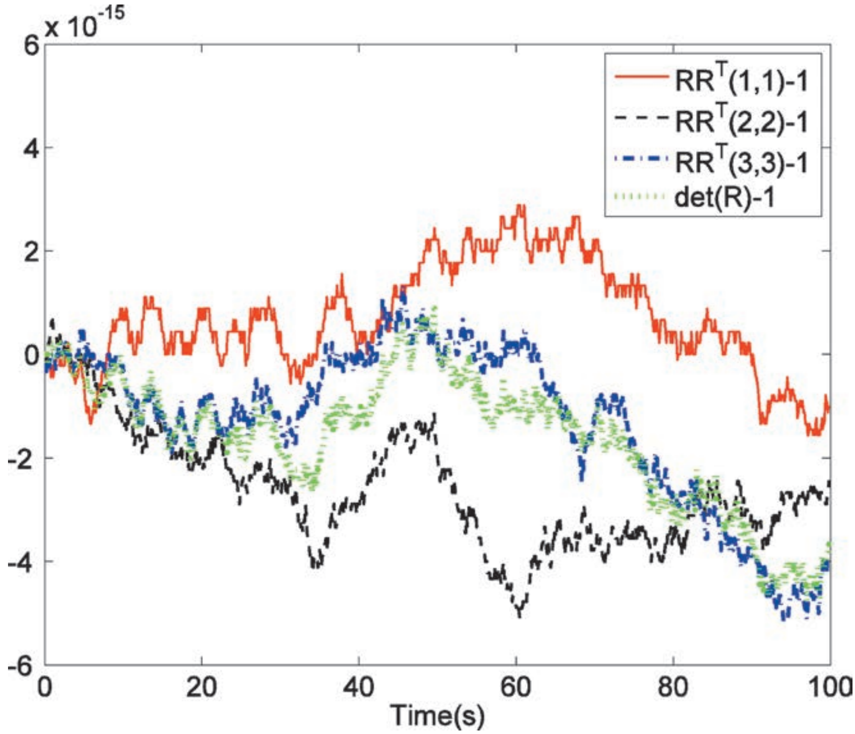

Fig. 2 – Reconstruction of the special orthogonal rotation matrix $\mathbf{R} \in SO(3)$. Diagonal elements of the product $\mathbf{R}^T\mathbf{R} = \boldsymbol{I}$ and determinant $\det \mathbf{R} = +1$.

In Figure 3, the coordinate of the body angular momentum calculated with the proposed scheme is compared to the RK-Munthe-Kaas scheme of the 4th order (where angular momentum equation is resolved in the vector space) and to the integration solution obtained with the one of the currently best second order Lie-group algorithms i.e. the method based on the Newmark vector scheme described in (Krysl, Endres, 2005). Since in the figures a drift from the constant analytical value is presented, it is visible that the proposed method yields better angular momentum preservation than the other tested numerical schemes (the other coordinates present similar comparative drift).

If, instead of integrating free-body rotation, the forcing terms have to be included, the equation (2) has the form

$$\dot{\mathbf{y}}(t) = \mathrm{ad}^*_{\boldsymbol{\omega}}\mathbf{y} + \mathbf{T}(t)\,, \tag{5}$$

where $\mathbf{T}$ is the body forcing torque. This means that we can also write

$$\dot{\mathbf{y}}(t) = \mathrm{dexp}_{-\tilde{\psi}(t)}(-\dot{\tilde{\psi}}(t))\mathbf{y}(t) + \mathbf{T}(t)\,, \tag{6}$$

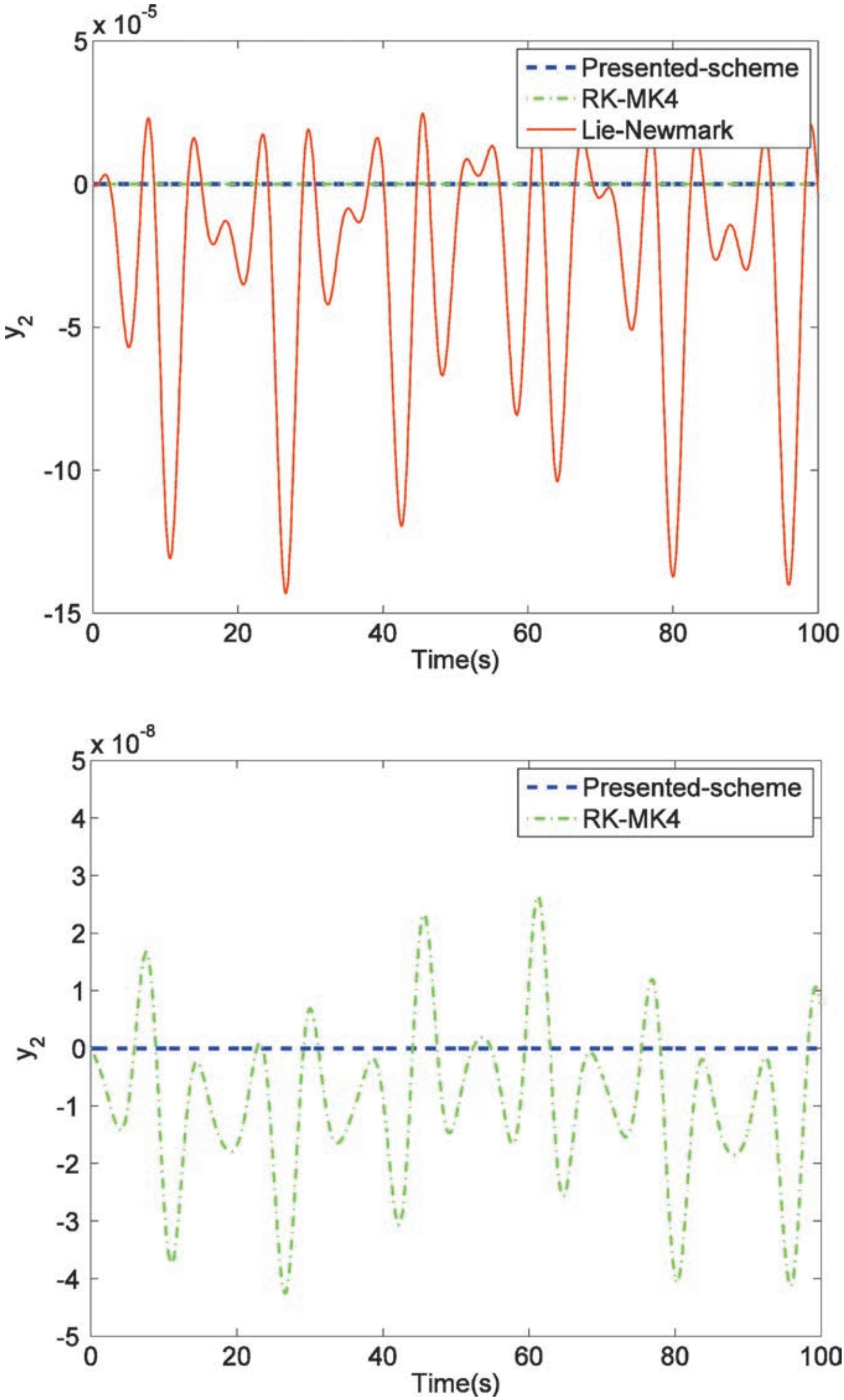

Fig. 3 – Angular momentum component drift – comparison of the integration methods.

and the angular momentum step upgrade has the form

$$\mathbf{y}_{n+1} = \mathrm{Ad}^{*}_{\exp(\tilde{\psi}(t))}(\mathbf{y}_n + \mathbf{R}_n^T \int_{t_n}^{t_{n+1}} \mathbf{R}(\tau)\mathbf{T}(\tau)d\tau), \tag{7}$$

while the RK-4 internal steps have to be modified accordingly. The described algorithm is easily incorporated in the overall MBS geometric integration scheme (1).

5. Conclusions

In the paper geometric scheme that extends coadjoint-preserving integration method for *SO*(3), and fully preserves angular momentum of the freely spinning rigid body, is presented.

Although the method is fully explicit, it generally outperforms tested semi-implicit 2nd order schemes (such as Lie-Newmark algorithm) or 4th order explicit RK-MK4 integration algorithm in terms of angular momentum preservation. In the presented form the scheme is focused on rotational rigid body dynamics (that is the most challenging part in the context of numerical integration of rigid body motion), but the method can be easily applied within the integration algorithms for dynamics of general rigid multibody systems.

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